Assignment 3.

This homework is due *Thursday*, September 24.

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% (but not over 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.3–2.4 in Bartle–Sherbert.

(1) [3pt] (2.3.4) let S be a a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$, and let $aS = \{as \mid s \in S\}$. Prove that
   $$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$
   (b) Let $b < 0$, and let $bS = \{bs \mid s \in S\}$. Prove that
   $$\inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.$$

(2) (a) [2pt] (Part of 2.3.11) Let $S \subset \mathbb{R}$ be a bounded set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \leq \sup S$.
   (b) [2pt] (2.3.10) Show that if A and B are bounded nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup A \cup B = \sup\{\sup A, \sup B\}$.
   (c) [2pt] (2.4.7) For A, B as in previous item, show that $A + B = \{a + b : a \in A, b \in B\}$ is a bounded set. Prove that $\sup(A+B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.
   (d) [2pt] Find $\sup\{\frac{1}{n} : n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} : n \in \mathbb{N}\}$, $\sup\{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}$. (*Hint:* for the last two questions, use the previous item 2c.)
   (e) [2pt] For A, B as in item 2c, show that $AB = \{ab : a \in A, b \in B\}$ is a bounded set. Is it true that always $\sup AB = \sup A \cdot \sup B$?

(3) [3pt] (2.4.8) Let X be a nonempty set, and let functions f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that
$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$$
and that
$$\inf\{f(x) + g(x) \mid x \in X\} \geq \inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\}.$$
Give examples to show that each of these inequalities can be either an equality or a strict inequality.

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(4) (2.4.9) Let $X = Y = (0,1) \subseteq \mathbb{R}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x,y) = 2x + y$.

   (a) [1pt] For each $x \in X$, find $f(x) = \sup\{h(x,y) \mid y \in Y\}$; then find $\inf\{f(x) \mid x \in X\}$.

   (b) [1pt] For each $y \in Y$, find $g(y) = \inf\{h(x,y) \mid x \in X\}$; then find $\sup\{g(y) \mid y \in Y\}$. Compare with the result found in (a).

   (c) [2pt] (2.4.10) Perform the computations in (a), (b) for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x,y) = \begin{cases} 0, & \text{if } x < y, \\ 1, & \text{if } x \geq y. \end{cases}$$

(5) [3pt] (2.4.11) Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by

$$f(x) = \sup\{h(x,y) \mid y \in Y\}, \qquad g(y) = \inf\{h(x,y) \mid x \in X\}.$$

Prove that $\sup\{g(y) \mid y \in Y\} \leq \inf\{f(x) \mid x \in X\}$.

Comment. This inequality can be also expressed in the following way:

$$\sup_{y \in Y} \inf_{x \in X} h(x,y) \leq \inf_{x \in X} \sup_{y \in Y} h(x,y).$$

Previous problem shows that this non-strict inequality may be either an equality or a strict inequality.